

## GAUSSIAN BRONZE FIBONACCI NUMBERS

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### ABSTRACT

In this paper, we extend the Bronze Fibonacci number to the Gaussian Bronze Fibonacci number. Moreover, we obtain Binet formula, generating function and some identities for this number.

**Key words:** Bronze Fibonacci numbers, Gaussian Bronze Fibonacci numbers, generating function, Binet formula

### 1. INTRODUCTION

Integer sequences appear often in many branches of science. The most famous example is the Fibonacci numbers that have been known for more than two thousand years and have application in mathematics, biology, physics, engineering, computer sciences, economics, architecture and so on.

Pell, Lucas and Jacobsthal numbers are the most studied numbers like Fibonacci numbers. These studies are about generalization, some properties of these numbers and defining the matrix sequence of them.

In [1], the authors generalized Fibonacci sequence and many properties were deduced from elementary matrix algebra. In [2], the authors obtained new identities for k-Fibonacci numbers and investigated divisibility properties of these numbers. Identities for the generalized second order recurrence relation by using the generalized Euler-Seidel matrix were obtained. As a consequence of this they gave some properties and generating functions of well-known integer sequences in [3].

Gaussian forms of these numbers have taken so much interest recently. In 1963, Horadam introduced the concept the complex Fibonacci number called as the Gaussian Fibonacci numbers [9]. In another study, two of the complex Fibonacci sequences are considered and the author extended some relationship which are known about the common Fibonacci sequences. Also the author gave many identities between the usual Fibonacci and Gaussian Fibonacci sequences [10]. Moreover, Horadam investigated the complex Fibonacci polynomials. In 1977, Berzsenyi aimed to present a natural manner of extension of the Fibonacci numbers into the complex plane. Then some interesting identities are obtained for usual Fibonacci numbers [6]. Also Gaussian Pell, Gaussian Pell-Lucas, Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas numbers were studied by many authors. In these studies, the authors gave the Binet formulas for these sequences then they obtained the generating functions and some identities involving the terms of these sequences [4,5].

In [10], Gaussian Fibonacci numbers are defined by the following recurrence relation

$$GF_n = GF_{n-1} + GF_{n-2}, n > 1$$

with initial conditions  $GF_0 = i, GF_1 = 1$ . Also it can be seen that

$$GF_n = F_n + iF_{n-1}$$

where  $F_n$  is the usual Fibonacci number. In the same study the Gaussian Lucas numbers are defined as

$$GL_n = GL_{n-1} + GL_{n-2}, n > 1$$

with initial conditions  $GL_0 = 2 - i, GL_1 = 1 + 2i$ . Also,

$$GL_n = L_n + iL_{n-1}$$

where  $L_n$  is the  $n$ th Lucas number. Then some relationships are extended for these numbers.

Gaussian Fibonacci numbers were found interesting and studied by many authors. In [9], the results in [8] were extended to complex Fibonacci numbers.

For  $n$  is an integer and  $m$  is a nonnegative integer Gaussian Fibonacci number  $GF_{n+mi}$   $F_{\{n+mi\}}$  is defined and some identities are studied in [6]. In another paper [7] complex Fibonacci numbers were defined and some identities for these numbers were obtained. Pethe and Horadam defined generalized Gaussian Fibonacci numbers and obtained some summation identities involving Fibonacci polynomials and Pell polynomials [11].

Bronze Fibonacci numbers are defined as

$$B_{n+2} = 3B_{n+1} + B_n$$

with the initial conditions  $B_0 = 0$  and  $B_1 = 1$  where  $n \geq 1$ .

In this study, we extend the Bronze Fibonacci number to the Gaussian Bronze Fibonacci number. Moreover, we obtain Binet formula, generating function and some identities for this number sequence.

## 2. MAIN RESULT

**Definition 1** For  $n \geq 1$  The Gaussian Bronze Fibonacci numbers are defined by the following recurrence relation

$$GB_{n+2} = 3GB_{n+1} + GB_n$$

with the initial conditions  $GB_0 = i, GB_1 = 1$ .

Also,  $GB_n = B_n + iB_{n-1}$  where  $B_n$  is the  $n$ th Bronze Fibonacci number.

**Theorem 2** The generating function of the Gaussian Bronze Fibonacci numbers is as in the following:

$$g(z) = \sum_{n=0}^{\infty} GB_n z^n = \frac{z + i(1 - 3z)}{1 - 3z - z^2}.$$

**Proof.** Let  $g(z)$  be the generating function of the Gaussian Bronze Fibonacci numbers then

$$\begin{aligned}
\sum_{n=0}^{\infty} GB_{n+2} z^n &= \sum_{n=0}^{\infty} 3GB_{n+1} z^n + \sum_{n=0}^{\infty} GB_n z^n \\
\sum_{n=2}^{\infty} GB_n z^{n-2} &= \sum_{n=1}^{\infty} 3GB_n z^{n-1} + \sum_{n=0}^{\infty} GB_n z^n \\
\frac{1}{z^2} \sum_{n=2}^{\infty} GB_n z^n &= \frac{1}{z} \sum_{n=1}^{\infty} 3GB_n z^n + \sum_{n=0}^{\infty} GB_n z^n \\
\frac{1}{z^2} (g(z) - i - z) &= \frac{1}{z} (3g(z) - 3i) + g(z) \\
g(z) \left( \frac{1}{z^2} - \frac{3}{z} - 1 \right) &= \frac{i+z}{z^2} - \frac{3i}{z}
\end{aligned}$$

after some calculations  $g(z)$  can be obtained as

$$g(z) = \frac{z+i(1-3z)}{1-3z-z^2}.$$

**Theorem 3** Binet formula for Gaussian Bronze Fibonacci numbers is given by

$$GB_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i \frac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta}$$

where  $\alpha, \beta$  are the roots of the characteristic equation.

**Proof.** From the theory of difference equations, it is known that the general term of Gaussian Bronze Fibonacci numbers is as in the following

$$GB_n = x_1 \alpha^n + x_2 \beta^n$$

where  $x_1$  and  $x_2$  are the coefficients. Using the values  $n = 0, 1$  the coefficients are obtained as

$$x_1 = \frac{1-i\beta}{\alpha-\beta} \text{ and } x_2 = \frac{-1+i\alpha}{\alpha-\beta}.$$

Considering the values  $x_1, x_2$  and making some calculations we obtain

$$GB_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i \frac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta}.$$

**Theorem 3** The sum of the Gaussian Bronze Fibonacci numbers is

$$\sum_{k=1}^n GB_k = \frac{1}{3} (GB_{n+1} + GB_n - (i+1)).$$

**Proof.** We can prove it by using the definition with usual Bronze Fibonacci number of this number sequence

$$\begin{aligned}
\sum_{k=1}^n GB_k &= \sum_{k=1}^n (B_k + iB_{k-1}) \\
&= \sum_{k=1}^n B_k + i \sum_{k=1}^n B_{k-1} \\
&= \sum_{k=1}^n B_k + i \sum_{k=1}^n B_k - iB_n \\
&= \frac{1}{3}(B_{n+1} + B_n - 1) + \frac{i}{3}(B_{n+1} + B_n - 1) - iB_n \\
&= \frac{1}{3}(B_{n+1} + iB_n + iB_{n+1} + B_n - (i+1)) - iB_n \\
&= \frac{1}{3}(B_{n+1} + iB_n + i(3B_n + B_{n-1}) + B_n - (i+1)) - iB_n \\
&= \frac{1}{3}(B_{n+1} + iB_n + B_n + iB_{n-1} - (i+1)) \\
&= \frac{1}{3}(GB_{n+1} + GB_n - (i+1)).
\end{aligned}$$

**Theorem 5** The sum of the even indexed Gaussian Bronze Fibonacci numbers can be given as

$$\sum_{k=1}^n GB_{2k} = \frac{1}{3}(GB_{2n+1} - 1).$$

**Proof.** It can be proved by using the definition of Gaussian Bronze Fibonacci numbers.

$$\begin{aligned}
\sum_{k=1}^n GB_{2k} &= \sum_{k=1}^n (B_{2k} + iB_{2k-1}) \\
&= \sum_{k=1}^n B_{2k} + i \left( \sum_{k=0}^n B_{2k+1} - B_{2n+1} \right) \\
&= \sum_{k=1}^n B_{2k} + i \sum_{k=0}^n B_{2k+1} - iB_{2n+1} \\
&= \frac{1}{3}(B_{2n+1} - 1) + \frac{i}{3}B_{2n+2} - iB_{2n+1} \\
&= \frac{1}{3}(B_{2n+1} + iB_{2n+2} - 1) - iB_{2n+1} \\
&= \frac{1}{3}(B_{2n+1} + i(3B_{2n+1} + B_{2n}) - 1) - iB_{2n+1} \\
&= \frac{1}{3}(B_{2n+1} + iB_{2n} - 1) \\
&= \frac{1}{3}(GB_{2n+1} - 1).
\end{aligned}$$

**Theorem 6** The sum of the odd indexed Gaussian Bronze Fibonacci numbers can be given as

$$\sum_{k=0}^n GB_{2k+1} = \frac{1}{3}(GB_{2n+2} - i).$$

**Proof.** It can be proved similarly by the previous theorem.

$$\begin{aligned}
\sum_{k=0}^n GB_{2k+1} &= \sum_{k=0}^n (B_{2k+1} + iB_{2k}) \\
&= \sum_{k=0}^n B_{2k+1} + i \sum_{k=0}^n B_{2k} \\
&= \frac{1}{3} (B_{2n+2} + iB_{2n+1} - i) \\
&= \frac{1}{3} (GB_{2n+2} - i).
\end{aligned}$$

**Theorem 7** The convolution identity is given as following two form

$$GB_{n+m} = GB_{n+1}B_m + GB_nB_{m-1} \quad (1)$$

or

$$GB_{n+m} = B_{n+1}GB_m + B_nGB_{m-1}. \quad (2)$$

**Proof.** We will use the definition of this number sequence again

$$GB_{n+m} = B_{n+m} + iB_{n+m-1}.$$

By using the convolution identity for usual Bronze Fibonacci number,  $GB_{n+m}$  is obtained as

$$\begin{aligned}
GB_{n+m} &= (B_{n+1}B_m + B_nB_{m-1}) + i(B_nB_m + B_{n-1}B_{m-1}) \\
&= (B_{n+1} + iB_n)B_m + (B_n + iB_{n-1})B_{m-1} \\
&= GB_{n+1}B_m + GB_nB_{m-1}.
\end{aligned}$$

The second form can be obtained similarly.

**Theorem 8** The  $2n$  th Gaussian Bronze Fibonacci number is

$$GB_{2n} = \frac{1}{3} (B_{n+1}^2 - B_{n-1}^2) + i(B_n^2 + B_{n-1}^2)$$

or

$$GB_{2n} = \frac{1}{3} (B_{n+1}^2 - B_{n-1}^2) + i(B_{n+1}B_{n-1} + B_nB_{n-2}).$$

**Proof.** If it is taken  $n = m$  in the equality (1), we have

$$\begin{aligned}
GB_{2n} &= GB_{n+1}B_n + GB_nB_{n-1} \\
&= (B_{n+1} + iB_n)B_n + (B_n + iB_{n-1})B_{n-1} \\
&= (B_{n+1} + B_{n-1})B_n + i(B_n^2 + B_{n-1}^2) \\
&= (B_{n+1} + B_{n-1}) \left( \frac{B_{n+1} - B_{n-1}}{3} \right) + i(B_n^2 + B_{n-1}^2) \\
&= \frac{1}{3} (B_{n+1}^2 - B_{n-1}^2) + i(B_n^2 + B_{n-1}^2).
\end{aligned}$$

If it is taken  $n = m$  in the equality (2), the second equality is obtained for  $GB_{2n}$ .

**Theorem 9** The odd Gaussian Bronze Fibonacci number is

$$GB_{2n+1} = (B_{n+1}^2 + B_n^2) + i \left( \frac{B_{n+1}^2 - B_{n-1}^2}{3} \right).$$

**Proof.** It can be proved by the help of convolution identity.

**Theorem 10** (Cassini Identity) For  $n \geq 1$ , we get

$$GB_{n+1}GB_{n-1} - GB_n^2 = (-1)^n (2 - 3i). \quad (3)$$

**Proof.** We will use the induction method on  $n$ . If  $n = 1$ , then

$$GB_2GB_0 - GB_1^2 = -2 + 3i.$$

Now, we suppose that the equation (3) holds for  $k$ . Then we show that the equation (3) holds for  $k+1$ .

$$\begin{aligned} GB_{k+2}GB_k - GB_{k+1}^2 &= (3GB_{k+1}GB_k) \left( \frac{1}{3}GB_{k+1} - \frac{1}{3}GB_{k-1} \right) - GB_{k+1}^2 \\ &= \frac{1}{3}GB_{k+1}GB_k - \frac{1}{3}GB_kGB_{k-1} - \left[ (-1)^k (2 - 3i) + GB_k^2 \right] \\ &= \frac{1}{3}GB_{k+1}GB_k - GB_k \left( \frac{1}{3}GB_{k-1} + GB_k \right) + (-1)^{k+1} (2 - 3i) \\ &= (-1)^{k+1} (2 - 3i) \end{aligned}$$

So, we have the conclusion.

**Theorem 11** (Catalan Identity) For positive integers  $n, k$  we have

$$GB_{n+k}GB_{n-k} - GB_n^2 = (-1)^n (2 - 3i) \left[ \frac{(-1)^{k-1} (\alpha^k + \beta^k)^2 + 2}{13} \right]. \quad (4)$$

**Proof.** By using the Binet formula, the proof can be easily seen.

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