EJONS International Journal on Mathematic, Engineering and Natural Sciences
ISSN 2602 - 4136

Article Arrival Date
30.01.2021
Article Type
Research Article
Article Published Date
20.03.2022
Doi Number: http://dx.doi.org/10.38063/ejons.551

APPROXIMATION PROPERTIES OF A KANTOROVICH TYPE OPERATOR

KANTOROVICH TİPLİ BİR OPERATÖRÜN YAKLAŞIM ÖZELLİKLERİ

Nazmiye GONUL BILGIN

Zonguldak Bulent Ecevit University, Faculty of Arts and Sciences, Department of Mathematics, 67100, Zonguldak, TURKEY.

ABSTRACT

This study has been prepared in the theory of approximation, which has an important place in the fields of application. In this paper, a modification of operators of the Gadjiev-Ibragimov type that preserves test functions will be described. The paper is about Kantorovich-type modification of a generalization of Gadjiev-Ibragimov operators. It is aimed to present a new materials to researchers who will conduct applied studies by examining the uniform convergence of this of the new operator in integral form, whose terms are functions defined on C[0,1].

Based on the Korovkin approximation theorem, properties of convergence for these operators and then some direct theorems will be given.The rate of convergence of these operators will be calculated with the help of the modulus of continuity by using the classical second order moments. By using the definition created by Ozarslan and Aktuglu, the approximation theorem for functions from the Lipschitz class will be given and the approximation properties of these modified operators in weighted spaces will also be examined. Also, properties of approximation will be demonstrated with graphics and numerical calculations using the Maple program.

Keywords: Kantorovich Operators, Gadjiev-Ibragimov Type Operators, Korovkin Theorems.

ÖZET

Bu çalışma, uygulama alanlarında önemli bir yere sahip olan yaklaşım teorisinde hazırlanmıştır. Bu çalışmada, test fonksiyonlarını koruyan Gadjiev-Ibragimov tipi operatörlerin bir modifikasyonu tanımlanacaktır. Makale, Gadjiev-Ibragimov operatörlerinin bir genellemesinin Kantorovich tipi modifikasyonu hakkındadır. Terimleri C[0,1] üzerinde tanımlanan fonksiyonlar olan bu yeni operatörün integral formda düzgün yakınsaklığı incelenerek; uygulamalı çalışmalar yapacak araştırmacılara yeni bir materyal sunulması amaçlanmaktadır.

Korovkin yaklaşım teoremine dayalı olarak, bu operatörler için yakınsama özellikleri ve ardından bazı doğrudan teoremler verilecektir. Bu operatörlerin yakınsama oranları, klasik ikinci dereceden momentler kullanılarak süreklilik modülü yardımıyla hesaplanacaktır. Özarslan ve Aktuğlu' nun oluşturduğu tanım kullanılarak Lipschitz sınıfından fonksiyonlar için yaklaşım teoremi verilerek bu modifiye operatörlerin ağırlıklı uzaylarda yaklaşım özellikleri de incelenecektir. Ayrıca, yakınsaklık özellikleri Maple programı kullanılarak grafikler ve sayısal hesaplamalar ile gösterilecektir.

Anahtar Kelimeler: Kantorovich Operatörleri, Gadjiev-Ibragimov Tipi Operatörler, Korovkin Teoremi.

1. INTRODUCTION

Approximation theory is concerned with the ability to approximate functions by simpler and more easily calculated functions. In recent years the number of branches related to approximation theory has been increasing steadily.

Accordingly, many generalizations of linear positive operators have been studied, one of them is the Kantorovich modification. The subject of Kantorovich operators is still the focus of many researchers. After defining a new operator, making Kantorovich generalizations of that operator is an expected modification by researchers following the studies in that field. For this purpose a generalization of the operator defined in (Gonul and Coskun, 2013) will be made in this sense and important approximation properties will be given now.

In 1970, the sequence of linear positive operators named after them was introduced in (Gadjiev and Ibragimov,1970). This operator is called Ibragimov-Gadjiev or Gadjiev-Ibragimov operators in literature. Here, the second expression will be used. After then, a lot of generalizations of Gadjiev-Ibragimov operators are studied for example in (Doğru, 1997), (Ispir et al, 2008), (Gonul and Coskun, 2012). Apart from these, Aral, proved that derivatives of Gadjiev-Ibragimov operators converges to derivatives of the functions in (Aral, 2005). Some convergence properties of Durrmeyer versions of these operators are defined in (Ulusoy et.al. 2015). In (Gonul Bilgin and Coskun, 2018), the approximation properties of Gadjiev-Ibragimov type operators are compared and then is gave main properties of a two dimensional version of these operators in (Gonul Bilgin and Ozgur, 2019). q-generalizations of the classical version of this operators are given in (Herdem and Buyukyazici 2018).

Here, the space of continuous functions defined on $[0,1]$; with $C[0,1]$ and the space of Lebesgue integrable functions defined in the same interval will be denoted by $L_1[0,1]$.

Let's remember the operator whose properties are given in (Gonul and Coskun, 2013).

$$L_n(f,x)=\sum_{\nu=0}^{\infty} f\left(\frac{\nu}{\beta_n}\right)\ K_{n,\nu}(x)\frac{(-\alpha_n)^{\nu}}{\nu!} \tag{3}$$

Lemma 1.1 For every $n=0,1,2,\dots$ and for all $x\in[0,1]$, $L_n(f,x)$ satisfy the following equalities:

$i)\ L_n(1,x)=1,$

$ii)\ L_n(t,x)=\dfrac{\alpha_n}{\beta_n}nx,$

$iii)\ L_n(t^2,x)=\dfrac{{\alpha_n}^2 n(n+m)x^2}{{\beta_n}^2}+\dfrac{\alpha_n nx}{{\beta_n}^2},$

$$iv)\ L_n(t^3,x)=\left\{\left(\frac{\alpha_n}{\beta_n}\right)^3 n^3+3\left(\frac{\alpha_n}{\beta_n}\right)^3 n^2m+2\left(\frac{\alpha_n}{\beta_n}\right)^3 nm^2\right\}x^3+\left\{\frac{3}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^2 n^2\right.$$

$$\left.+\frac{3}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^2 nm\right\}x^2+\frac{1}{{\beta_n}^2}\left(\frac{\alpha_n}{\beta_n}\right) nx\,,$$

$$v)\ L_n(t^4,x)=\{\left(\frac{\alpha_n}{\beta_n}\right)^4 n^4+6\left(\frac{\alpha_n}{\beta_n}\right)^4 n^3m+11\left(\frac{\alpha_n}{\beta_n}\right)^4 n^2m^2+6\left(\frac{\alpha_n}{\beta_n}\right)^4 nm^3\}x^4$$

$$+\left\{\frac{6}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 n^3+\frac{18}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 n^2m+\frac{12}{\beta_n}\left(\frac{\alpha_n}{\beta_n}\right)^3 nm^2\right\}x^3$$

$$+\left\{\frac{7}{{\beta_n}^2}\left(\frac{\alpha_n}{\beta_n}\right)^2 n^2+\frac{7}{{\beta_n}^2}\left(\frac{\alpha_n}{\beta_n}\right)^2 nm\right\}x^2+\frac{1}{{\beta_n}^3}\left(\frac{\alpha_n}{\beta_n}\right) nx.$$

The design of the next sections is as follows. First of all, the Kantorovich generalization of the (3) operator will be defined and the equations related to the test functions will be given. After it has been shown that this operator satisfies a Korovkin type theorem, the approximation theorem for weighted spaces will be proved by using the results obtained in the previous section. In the fourth chapter, rate of convergence using modulus of continuity and graphical representation of the approximation will be made.

2. METHODS AND MATERIALS

In this section, firstly, the operator will be defined and the Korovkin type approximation theorem will be proved for this operator.

Definition 2.1 For $(c_n), (d_n)$; let $\lim_{n\to\infty} d_n = \infty$, $\lim_{n\to\infty} \frac{c_n}{d_n} = 0$ and $\lim_{n\to\infty} n\frac{c_n}{d_n} = 1$. In this case, a modification of Kantorovich-type Gadjiev-Ibragimov operators, give as follows.

$$L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f,x) = d_n \sum_{r=0}^{\infty} N_{n,r}(x) \frac{(-c_n)^r}{r!} \int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}} f(u)du \tag{4}$$

Here $N_{n,r}(x)$ is the function depending on the parameters r and n to get the following conditions:

i) For every $n, r = 0, 1, 2, \ldots$ and for all $x \in [0,1]$

$(-1)^r N_{n,r}(x) \geq 0.$

ii) For all $x \in [0,1]$,

$$\sum_{r=0}^{\infty} N_{n,r}(x) \frac{(-c_n)^r}{r!} = 1.$$

iii) $N_{n,r}(x) = -nxN_{n+p,r-1}(x)$,

for any $x \in [0,1]$ where $n + p$ is natural number and p is a constant independent of r.

Lemma 2.1

The following equations are valid for the operator $L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f,x)$;

i) $L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(1,x) = 1$,

ii) $L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t,x) = \frac{1+2nxc_n}{2d_n}$,

iii) $L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t^2,x) = \frac{{c_n}^2 n(n+p)x^2}{{d_n}^2} + 2\frac{c_n nx}{{d_n}^2} + \frac{1}{3{d_n}^2}$.

Proof

From definition of operators

$L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(1,x) = 1$

can be obtain easily.

Then, for $L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t,x)$, since $(n+p) \in \mathbb{N}$,

$$L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t,x) = d_n \sum_{r=0}^{\infty} N_{n,r}(x) \frac{(-c_n)^r}{r!} \int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}} udu$$

$$= \frac{d_n}{2} \sum_{r=0}^{\infty} N_{n,r}(x) \frac{(-c_n)^r}{r!} \left[\left(\frac{r+1}{d_n} \right)^2 - \left(\frac{r}{d_n} \right)^2 \right]$$

$$= \sum_{r=0}^{\infty} \frac{r}{d_n} N_{n,r}(x) \frac{(-c_n)^r}{r!} + \frac{1}{2d_n} \sum_{r=0}^{\infty} N_{n,r}(x) \frac{(-c_n)^r}{r!}$$

$$= \frac{1 + 2nxc_n}{2d_n}.$$

Similarly, using definition of operators

$$L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t^2, x) = d_n \sum_{r=0}^{\infty} N_{n,r}(x) \frac{(-c_n)^r}{r!} \int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}} u^2 du$$

$$= \frac{d_n}{3} \sum_{r=0}^{\infty} N_{n,r}(x) \frac{(-c_n)^r}{r!} \left[\left(\frac{r+1}{d_n} \right)^3 - \left(\frac{r}{d_n} \right)^3 \right]$$

$$= \frac{d_n}{3} \sum_{r=0}^{\infty} N_{n,r}(x) \frac{(-c_n)^r}{r!} \left[\frac{3r^2 + 3r + 1}{{d_n}^3} \right]$$

$$= \sum_{r=0}^{\infty} \left(\frac{r}{d_n} \right)^2 N_{n,r}(x) \frac{(-c_n)^r}{r!} + \frac{1}{d_n} \sum_{r=0}^{\infty} \frac{r}{d_n} N_{n,r}(x) \frac{(-c_n)^r}{r!} + \frac{1}{3{d_n}^2} \sum_{r=0}^{\infty} N_{n,r}(x) \frac{(-c_n)^r}{r!}$$

$$= L_n(t^2, x) + \frac{1}{d_n} L_n(t, x) + \frac{1}{3{d_n}^2} L_n(1, x)$$

$$= \frac{{c_n}^2 n(n+p)x^2}{{d_n}^2} + 2\frac{c_n nx}{{d_n}^2} + \frac{1}{3{d_n}^2}$$

is obtained.

Now will be given the following basic properties needed to work on the main results.

Lemma 2.2

The $L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f, x)$ satisfy the following equations;

$i) \lim_{n\to\infty} \left\| L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(1, x) - 1 \right\|_{C[0,1]} = 0$

$ii) \lim_{n\to\infty} \left\| L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t, x) - x \right\|_{C[0,1]} = 0$

$iii) \lim_{n\to\infty} \left\| L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t^2, x) - x^2 \right\|_{C[0,1]} = 0$

Proof

Using Lemma 2.1,

$$\lim_{n\to\infty} \left\| L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(1, x) - 1 \right\|_{C[0,1]} = 0$$

can be written. Definition of (c_n) , (d_n) and if Lemma 2.1 ii) is used again

$$\left| L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t, x) - x \right| = \left| \frac{1 + 2nxc_n}{2d_n} - x \right| = \left| x\left(\frac{nc_n}{d_n} - 1 \right) + \frac{1}{2d_n} \right|$$

is written. So for all $x \in [0,1]$ and using $\lim_{n\to\infty} n\,\frac{c_n}{d_n} = 1$;

$$\max_{x\in[0,1]}\left|x\left(\frac{nc_n}{d_n}-1\right)+\frac{1}{2d_n}\right| \le \left|\frac{nc_n}{d_n}-1\right|+\left|\frac{1}{2d_n}\right|$$

is obtained. Then,

$$\lim_{n\to\infty}\left\|L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t,x)-x\right\|_{C[0,1]} \le \lim_{n\to\infty}\left|\frac{nc_n}{d_n}-1\right| + \lim_{n\to\infty}\left|\frac{1}{2d_n}\right| = 0$$

is geting.

Finally, the next equation from the definition of $L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t^2,x)$ is valid.

$$\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t^2,x)-x^2\right| = \left|\left[\frac{{c_n}^2 n(n+p)x^2}{{d_n}^2}+2\frac{c_n nx}{{d_n}^2}+\frac{1}{3{d_n}^2}\right]-x^2\right|$$

$$=\left|x^2\left(\frac{{c_n}^2}{{d_n}^2}n(n+p)-1\right)+2n\frac{c_n}{{d_n}^2}x+\frac{1}{3{d_n}^2}\right|.$$

By definition of the norm in the studied space,

$$\max_{x\in[0,1]}\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t^2,x)-x^2\right| = \max_{x\in[0,1]}\left|x^2\left(\frac{{c_n}^2}{{d_n}^2}n(n+p)-1\right)+2n\frac{c_n}{{d_n}^2}x+\frac{1}{3{d_n}^2}\right|$$

$$\le\left|\left(\frac{{c_n}^2}{{d_n}^2}n(n+p)-1\right)\right|+\left|2n\frac{c_n}{{d_n}^2}\right|+\left|\frac{1}{3{d_n}^2}\right|$$

is obtained. Definition of (c_n), (d_n) and since $(n+p)\in\mathbb{N}$,

$$\lim_{n\to\infty}\left\|L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t^2,x)-x^2\right\|_{C[0,1]}=0$$

is shown.

Theorem 2.1

Let $f \in C[0,1]$, then for all $x \in [0,1]$

$$\lim_{n\to\infty}\left\|L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(f,x)-f(x)\right\|_{C[0,1]}=0.$$

The proof is easily obtained from the above Lemma 2.2 and Korovkin's theorem.

Lemma 2.3

Let $j = 0,1,2$ and j-th degree moment for the $L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(f,x)$, is defined with

$$\mathcal{M}_{n,j}(x) = L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}\left((t-x)^j,x\right),$$

then,

$$\mathcal{M}_{n,0}(x)=1,$$

$$\mathcal{M}_{n,1}(x)=x\left(\frac{1+2nxc_n}{2d_n}-1\right)+\frac{1}{2d_n}$$

$$\mathcal{M}_{n,2}(x)=\left(\frac{{c_n}^2 n(n+p)}{{d_n}^2}-\frac{nc_n}{d_n}+1\right)x^2+\left(2\frac{c_n n}{{d_n}^2}-\frac{1}{2d_n}\right)x+\frac{1}{3{d_n}^2}$$

is obtained.

Proof

$t-x$ substituting in the operator;

$$L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t-x,x)=d_n\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}(u-x)du$$

$$=d_n\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}(u-x)du$$

$$=d_n\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\left[\frac{1}{2}\left(\frac{r+1}{d_n}\right)^2-\left(\frac{r}{d_n}\right)^2\right]-d_n\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\left[\left(\frac{r+1}{d_n}\right)x-\left(\frac{r}{d_n}\right)x\right]$$

$$=\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\frac{1}{2d_n}(2r+1)-x\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}$$

$$=\sum_{r=0}^{\infty}\frac{r}{d_n}N_{n,r}(x)\frac{(-c_n)^r}{r!}+\frac{1}{2d_n}\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}-x$$

$$=x\left(\frac{1+2nxc_n}{2d_n}-1\right)+\frac{1}{2d_n}$$

is obtained.

Using Lemma 2.1,

$$L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}((t-x)^2,x)=d_n\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}(u-x)^2du$$

$$=d_n\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}(u-x)^2du$$

$$=d_n\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}(u^2-2ux+x^2)du$$

$$=\frac{d_n}{3}\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\left[\left(\frac{r+1}{d_n}\right)^3-\left(\frac{r}{d_n}\right)^3\right]$$

$$-xd_n\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\left[\left(\frac{r+1}{d_n}\right)^2-\left(\frac{r}{d_n}\right)^2\right]$$

$$+x^2\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}$$

$$=\left[\frac{{c_n}^2n(n+p)x^2}{{d_n}^2}+\frac{c_nnx}{{d_n}^2}\right]+\frac{1}{d_n}\left[\frac{c_nnx}{d_n}\right]+\frac{1}{3{d_n}^2}-x\left[\frac{1+2nxc_n}{2d_n}\right]+x^2$$

$$=\left(\frac{{c_n}^2n(n+p)}{{d_n}^2}-\frac{nc_n}{d_n}+1\right)x^2+\left(2\frac{c_nn}{{d_n}^2}-\frac{1}{2d_n}\right)x+\frac{1}{3{d_n}^2}$$

is written Thus, the proof is complete.

3. RESULTS

In this section, an important theorem will be given using the findings obtained in the previous section.

Lemma 3.1

For $L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t^3,x)$ and $L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t^4,x)$, the following equations are valid.

$$L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t^3,x)=\left\{\left(\frac{c_n}{d_n}\right)^3 n^3+3\left(\frac{c_n}{d_n}\right)^3 n^2p+2\left(\frac{c_n}{d_n}\right)^3 np^2\right\}x^3$$

$$+\left\{\frac{3}{d_n}\left(\frac{c_n}{d_n}\right)^2 n^2+\frac{3}{d_n}\left(\frac{c_n}{d_n}\right)^2 np+\frac{2}{3}\left(\frac{c_n}{d_n}\right)^2\frac{n(n+p)}{d_n}\right\}x^2$$

$$+\left(\frac{5}{3}\frac{1}{{d_n}^2}\left(\frac{nc_n}{d_n}\right)+\frac{1}{d_n}\left(\frac{nc_n}{d_n}\right)\right)x+\frac{1}{4{d_n}^3},$$

$$L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t^4,x)=\left\{\left(\frac{c_n}{d_n}\right)^4 n^4+6\left(\frac{c_n}{d_n}\right)^4 n^3p+11\left(\frac{c_n}{d_n}\right)^4 n^2p^2+6\left(\frac{c_n}{d_n}\right)^4 np^3\right\}x^4$$

$$+\left\{\frac{8}{d_n}\left(\frac{c_n}{d_n}\right)^3 n^3+\frac{24}{d_n}\left(\frac{c_n}{d_n}\right)^3 n^2p+\frac{16}{d_n}\left(\frac{c_n}{d_n}\right)^3 np^2\right\}x^3$$

$$+\left\{\frac{13}{{d_n}^2}\left(\frac{c_n}{d_n}\right)^2 n^2+\frac{13}{{d_n}^2}\left(\frac{c_n}{d_n}\right)^2 np+\frac{2}{{d_n}^2}\frac{{c_n}^2 n(n+p)}{{d_n}^2}\right\}x^2$$

$$+\left\{\frac{4}{{d_n}^3}\left(\frac{c_n}{d_n}\right)n+\frac{2}{{d_n}^2}\frac{c_n n}{{d_n}^2}\right\}x+\frac{1}{5{d_n}^4}.$$

Proof

Using the definition of the operator and the following equation

$s^3=s(s-1)(s-2)+3s^2-2s,$

$$L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t^3,x)=d_n\sum_{r=0}^{\infty}N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}u^3du$$

$$=\frac{d_n}{4}\sum_{r=0}^{\infty}N_{n,r}(x)\frac{(-c_n)^r}{r!}\left[\left(\frac{r+1}{d_n}\right)^4-\left(\frac{r}{d_n}\right)^4\right]$$

$$=\frac{1}{4{d_n}^3}\sum_{r=0}^{\infty}N_{n,r}(x)\frac{(-c_n)^r}{r!}[4r^3+6r^2+4r+1]$$

$$=L_n\left(\left(\frac{r}{d_n}\right)^3,x\right)+\frac{2}{3d_n}L_n\left(\left(\frac{r}{d_n}\right)^2,x\right)+\frac{1}{d_n}L_n\left(\frac{r}{d_n},x\right)+\frac{1}{4{d_n}^3}L_n(1,x).$$

$$=L_n(t^3,x)+\frac{2}{3d_n}L_n(t^2,x)+\frac{1}{d_n}L_n(t,x)+\frac{1}{4{d_n}^3}L_n(1,x).$$

From the Lemma 1.1 the desired equality is shown. Similarly;

$$L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t^4,x)=d_n\sum_{r=0}^{\infty}N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}u^4du$$

$$= \frac{d_n}{5}\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\left[\left(\frac{r+1}{d_n}\right)^5 - \left(\frac{r}{d_n}\right)^5\right]$$

$$= \frac{1}{5{d_n}^4}\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}[5r^4+10r^3+10r^2+5r+1]$$

$$= L_n(t^4,x)+\frac{2}{d_n}L_n(t^3,x)+\frac{2}{{d_n}^2}L_n(t^2,x)+\frac{1}{{d_n}^3}L_n(t,x)+\frac{1}{5{d_n}^4}L_n(1,x)$$

$$= \left\{\left(\frac{c_n}{d_n}\right)^4 n^4 + 6\left(\frac{c_n}{d_n}\right)^4 n^3p + 11\left(\frac{c_n}{d_n}\right)^4 n^2p^2 + 6\left(\frac{c_n}{d_n}\right)^4 np^3\right\}x^4$$

$$+\left\{\frac{8}{d_n}\left(\frac{c_n}{d_n}\right)^3 n^3 + \frac{24}{d_n}\left(\frac{c_n}{d_n}\right)^3 n^2p + \frac{16}{d_n}\left(\frac{c_n}{d_n}\right)^3 np^2\right\}x^3$$

$$+\left\{\frac{13}{{d_n}^2}\left(\frac{c_n}{d_n}\right)^2 n^2 + \frac{13}{{d_n}^2}\left(\frac{c_n}{d_n}\right)^2 np + \frac{2}{{d_n}^2}\frac{{c_n}^2 n(n+p)}{{d_n}^2}\right\}x^2$$

$$+\left\{\frac{4}{{d_n}^3}\left(\frac{c_n}{d_n}\right) n + \frac{2}{{d_n}^2}\frac{c_n n}{{d_n}^2}\right\}x + \frac{1}{5{d_n}^4}.$$

Thus, the proof is complete.

Now; the following well-known weighted spaces of functions which are defined on the $(0,\infty]$is considered. Let $\rho(x)=1+x^2$ weighted functions, $K_f>0$ be a positive constant depending of f. In the theorem that will be given from now on, $B_\rho([0,\infty))$, $C_\rho([0,\infty))$, $C_\rho^k([0,\infty))$ notations, which are defined in the literature as follows, will be used. The norm in this space is defined as 46

$$\|f\|_\rho = \sup_{[0,\infty)}\frac{|f(x)|}{\rho(x)}.$$

$$B_\rho([0,\infty)) := \{f:[0,\infty)\to\mathbb{R} : |f(x)|\le K_f\rho(x)\},$$

$$C_\rho([0,\infty)) := \{f\in B_\rho(([0,\infty)) : f \text{ continuous}\}$$

$$C_\rho^k([0,\infty)) := \left\{f\in C_\rho([0,\infty)) : \lim_{|x|\to\infty}\frac{f(x)}{\rho(x)} = k_f < \infty\right\}$$

Theorem 3.1

Let $f\in C_\rho^k((0,\infty])$ and $f\in L_1[0,1]$. In this case,

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(f,x)-f(x)\right|}{1+x^2}=0.$$

Proof

The proof will be made from Lemma 2.2, if the equation

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{J}}(t^m,x)-x^m\right|}{1+x^2}=0, \qquad (\text{for } m=0,1,2)$$

is shown, the proof is done. It is clear that

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(1,x)-1\right|}{1+x^2}=0.$$

For $m=1$

$$\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t,x)-x\right|=\left|x\left(\frac{nc_n}{d_n}-1\right)+\frac{1}{2d_n}\right|$$

and so,

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t,x)-x\right|}{1+x^2}\le \lim_{n\to\infty}\left|\frac{nc_n}{d_n}-1\right|+\lim_{n\to\infty}\left|\frac{1}{2d_n}\right|.$$

Thus

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t,x)-x\right|}{1+x^2}=0.$$

Finally, for t^2 using Lemma 2.1

$$\sup_{x\in[0,\infty)} \frac{\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t^2,x)-x^2\right|}{1+x^2}\le \sup_{x\in[0,\infty)}\left|\frac{x^2}{1+x^2}\right|\left|\frac{{c_n}^2 n(n+p)}{{d_n}^2}-1\right|$$

$$+\sup_{x\in[0,\infty)}\left|\frac{x}{1+x^2}\right|\left[2\frac{c_n n}{{d_n}^2}\right]+\sup_{x\in[0,\infty)}\left|\frac{1}{1+x^2}\right|\frac{1}{3{d_n}^2}$$

$$\le\left|\frac{c_n n}{d_n}\frac{c_n(n+p)}{d_n}-1\right|+\left[2\frac{c_n n}{{d_n}^2}\right]+\frac{1}{3{d_n}^2}.$$

From the properties of (c_n), (d_n);

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(t^2,x)-x^2\right|}{1+x^2}=0$$

is getting. So, every $f\in C_\rho^k\big((0,\infty]\big)$

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f,x)-f(x)\right|}{1+x^2}=0$$

is obtained.

4. DISCUSSION

In this section, calculation of modulus of continuity, approximation properties for functions from the Lipschitz class and the graphical representation of the approximation will be made.

Theorem 4.1 Let $f\in C[0,1]$ then the inequality

$$\left\|L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f,x)-f(x)\right\|_{C[0,1]}\le M\omega\left(f,\sqrt{\left(n\frac{c_n}{d_n}-1\right)^2+\frac{1}{d_n}}\right)$$

holds for sufficiently large n, where M is a constant independent of n.

$$\left|L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f,x)-f(x)\right|\le d_n\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}|f(u)-f(x)|du$$

$$\leq d_n \sum_{r=0}^{\infty} \omega(f,\delta_n) N_{n,r}(x) \frac{(-c_n)^r}{r!} \int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}} \left(1+\frac{|u-x|}{\delta_n}\right) du$$

$$\leq d_n \omega(f,\delta_n) \left\{ \frac{1}{\delta_n} \left[\sum_{r=0}^{\infty} N_{n,r}(x) \frac{(-c_n)^r}{r!} \int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}} |u-x|^2 du \right]^{1/2} + 1 \right\}$$

$$\leq \omega(f,\delta_n) \left\{ \frac{1}{\delta_n} \left(\left(\frac{{c_n}^2 n(n+p)}{{d_n}^2} - \frac{nc_n}{d_n} + 1 \right) x^2 + \left(2\frac{c_n n}{{d_n}^2} - \frac{1}{2d_n} \right) x + \frac{1}{3{d_n}^2} \right)^{1/2} + 1 \right\}$$

$$= \omega(f,\delta_n) \left\{ \frac{1}{\delta_n} \left(\left(\frac{{c_n}^2 n^2}{{d_n}^2} + \frac{{c_n}^2 np}{{d_n}^2} - \frac{nc_n}{d_n} + 1 \right) + \left(2\frac{c_n n}{{d_n}^2} - \frac{1}{2d_n} \right) + \frac{1}{3{d_n}^2} \right)^{1/2} + 1 \right\}$$

$$\leq \omega(f,\delta_n) \left\{ \frac{1}{\delta_n} \left[\left(n\frac{c_n}{d_n} - 1 \right)^2 + \left(2\frac{c_n n}{{d_n}^2} - \frac{1}{2d_n} \right) + \frac{1}{3d_n} \right]^{1/2} + 1 \right\}$$

$$\leq \omega(f,\delta_n) \left\{ \frac{1}{\delta_n} \left[\left(n\frac{c_n}{d_n} - 1 \right)^2 + \left(\frac{23}{6} \frac{1}{d_n} \right) \right]^{1/2} + 1 \right\}$$

$$\leq 4\omega(f,\delta_n) \left\{ \frac{1}{\delta_n} \left[\left(n\frac{c_n}{d_n} - 1 \right)^2 + \frac{1}{d_n} \right]^{1/2} + 1 \right\}.$$

It was shown by this theorem that the approximation is at $\sqrt{\left(n\frac{c_n}{d_n}-1\right)^2 + \frac{1}{d_n}}$ speed and this speed can be increased according to the choice of (c_n) and (d_n).

In the definition below; The space of functions from the Lipcitz class given by (Ozarslan and Aktuglu 2013) will be recalled, which will be used for the calculation of the operator's degree of approximation.

Let $\alpha_1 \geq 0, \alpha_2 > 0$ and $\gamma \in (0,1]$. For K which is a positive constant, the Lipschitz-type space, defined using two parameters, is represented as follows:

$$Lip_K^{\alpha_1,\alpha_2}(\gamma) = \left\{ f \in C[0,1]: |f(u)-f(x)| \leq K \frac{|u-x|^\gamma}{(u+\alpha_1 x^2+\alpha_2 x)^{\frac{\gamma}{2}}} : u \in [0,1], x \in (0,1] \right\}.$$

Theorem 4.2 Let $x \in (0,1]$ and $f \in Lip_K^{\alpha_1,\alpha_2}(\gamma)$ then

$$\left\| L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f,x) - f(x) \right\|_{C[0,1]} \leq K\omega \left(\frac{\mathcal{M}_{n,2}(x)}{\alpha_1 x^2 + \alpha_2 x} \right)^{\frac{\gamma}{2}}$$

is holds.

Proof

Let $\alpha_1 \geq 0, \alpha_2 > 0$ and $\gamma \in (0,1]$. From the Holder's inequality

$$\left| L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f,x) - f(x) \right| \leq d_n \sum_{r=0}^{\infty} N_{n,r}(x) \frac{(-c_n)^r}{r!} \left(\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}} |f(u)-f(x)|^{\frac{2}{\gamma}} du \right)^{\frac{\gamma}{2}}$$

$$\leq d_n\left(\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}|f(u)-f(x)|^{\frac{2}{\gamma}}du\right)^{\frac{\gamma}{2}}$$

$$\leq Md_n\left(\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}\frac{|u-x|^2}{(u+\alpha_1x^2+\alpha_2x)}du\right)^{\frac{\gamma}{2}}$$

$$\leq \frac{M}{(\alpha_1x^2+\alpha_2x)^{\frac{\gamma}{2}}}d_n\left(\sum_{r=0}^{\infty} N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}|u-x|^2du\right)^{\frac{\gamma}{2}}$$

$$=\frac{M}{(\alpha_1x^2+\alpha_2x)^{\frac{\gamma}{2}}}\left(\mathcal{M}_{n,2}(x)\right)^{\frac{\gamma}{2}}$$

this is the desired result.

A few applications will now be given to the theoretical study of structures. In the first two examples, the graphs related to the approach will be given, and in the third and fourth examples, the approach speed calculation that changes according to the sequence selection will be given.

Example 4.1

$$L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f,x)=d_n\sum_{r=0}^{m} N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}}f(u)du$$

$n=20$, $m=30$ and for $x\in[0,1]$ let $K_{n,\vartheta}(x)=(-1)^{\vartheta}(nx)^{\vartheta}e^{-nx\alpha_n}$, $(c_n)=1$ and $(d_n)=\sqrt{n}$. In this case, the graph of the operator's approximation to the $f(x)=\frac{1}{e^{x+1}+1}$ is given in Figure 4.1.

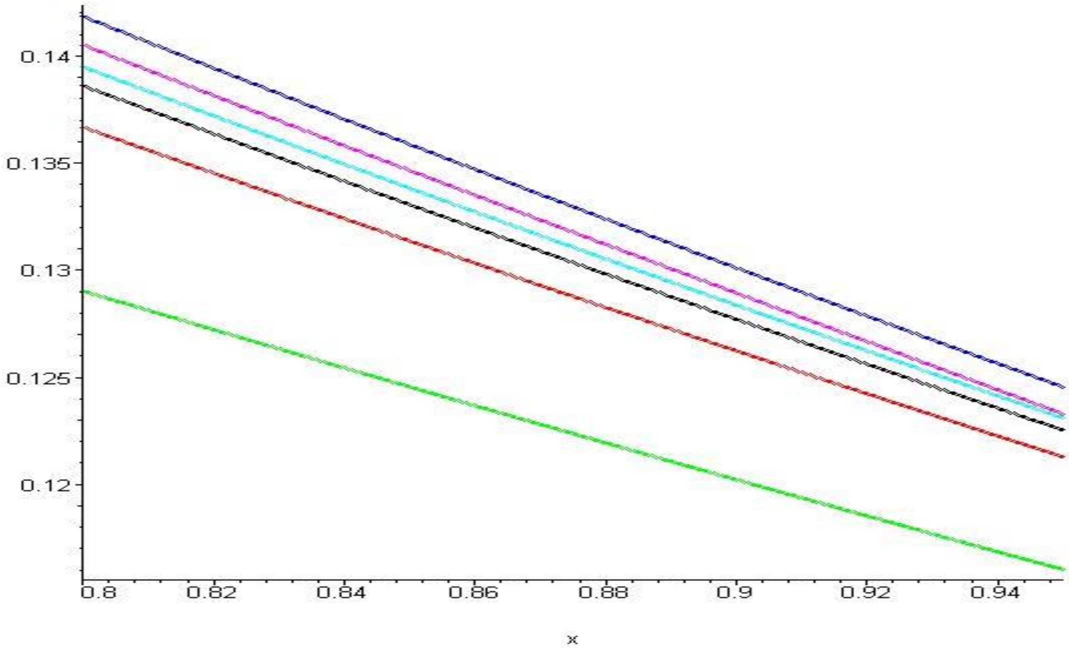

Figure 4.1 Approximation to the $f(x)=\frac{1}{e^{x+1}+1}$

Example 4.2

$$L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f,x) = d_n \sum_{r=0}^{m} N_{n,r}(x)\frac{(-c_n)^r}{r!}\int_{\frac{r}{d_n}}^{\frac{r+1}{d_n}} f(u)du$$

$n = 15$, $m = 20$ and for $x \in [0,1]$, let $K_{n,\vartheta}(x) = (-1)^{\vartheta}(nx)^{\vartheta}e^{-nx\alpha_n}$, $(c_n) = 1$ and $(d_n) = \sqrt{n}$. In this case, the graph of the operator's approximation to the $f(x) = \frac{1}{8}\cos(x^2+1)(x^2+1)+1$ is given in Figure 4.2.

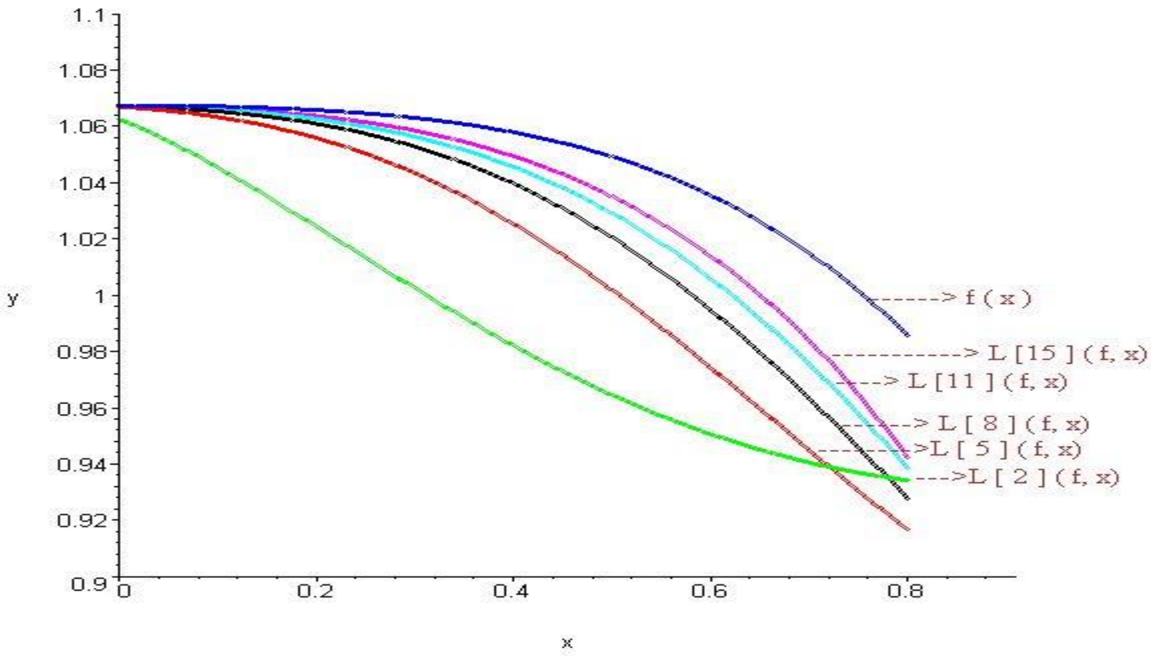

Figure 4.2 Approximation to the $f(x) = \frac{1}{8}\cos(x^2+1)(x^2+1)+1$

Example 4.3

Let $f(x) = \frac{2}{e^{(2x+5)}+1}$, $(c_n) = 1$ and $(d_n) = n$.Then, rate of convergence of approximation of operators to function is given Table 4.1.

Table 4.1. The error bound of function $f(x) = \frac{2}{e^{(2x+5)}+1}$ for $(c_n) = 1$ and $(d_n) = n$

n	**Error estimate of $f(x) = \frac{2}{e^{(2x+5)}+1}$ with $L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f,x)$**
10	**0.0250067657800**
10^2	**0.0096524161920**
10^3	**0.0032608562600**
10^4	**0.0010532623210**
10^5	**0.0003353207400**
10^6	**0.0001062640120**
10^7	**0.00003362633201**
10^8	**0.00001063584122**
10^9	**$0.3363545812.\ 10^{-5}$**

Example 4.4

For $f(x)=\frac{2}{e^{(2x+5)}+1}$, let taking $(c_n)=n$ and $(d_n)=n^2$.Then, rate of convergence of approximation of operators to function is given Table 4.2.

Table 4.2. The error bound of function for $(c_n)=n$ and $(d_n)=n^2$

n	**Error estimate of** $f(x)=\frac{2}{e^{(2x+5)}+1}$ **with** $L_n^{\mathcal{K},\mathcal{G},\mathcal{I}}(f,x)$
10	**0.00965241619200**
10^2	**0.001053262321000**
10^3	**0.000106264012000**
10^4	**0.000010635841220**
10^5	**0.1063667936 10^{-5}**
10^6	**0.1063686954$10^{-6}$**
10^7	**0.1063688856$10^{-7}$**
10^8	**0.1063689046$10^{-8}$**
10^9	**0.3363545812. 10^{-9}**

4. CONCLUSION

This Kantorovich type modified operator is a useful operator for approximating to functions. The operator also showed appropriate approximation properties for functions from the Lipchitz class. Since the theoretical work is supported by graphical and numerical calculations, it will guide researchers who will make Kantorovich modifications of different operators. In the next study, the idea was formed to examine the different approximation properties of the two-dimensional version of this operator.

5.ACKNOWLEDGMENT

This work is dedicated to Akif Gadjiev.

References

Aral, A. (2005). The Kantorovich Form of Ibragimov-Gadjiev Operators. Proceedings of the 8th WSEAS International Conference on Applied Mathematics, Tenerife, Spain, December 16-18, 2005 205-210.

Coşkun, T. (2012). On the order of weighted approximation by positive linear operators. Turkish Journal of Mathematics, 36(1), 113-120.

Doğru, O. (1997). On a certain family of linear positive operators. Turkish Journal of Mathematics, 21(4), 387-399.

Ibragimov, I. I., & Gadjiev, A. D. (1970). On a certain Family of linear positive operators. Soviet Math. Dokl., English Trans, 11, 1092-1095.

Gonul, N., & Coskun, E. (2012). Weighted approximation by positive linear operators. Proceedings of IMM of NAS of Azerbaijan, 36, 41-50.

Gonul, N., & Coskun, E. (2013). Approximation with modified Gadjiev-Ibragimov operators in C [0, A]. Journal of Computational Analysis & Applications, 15(1).

Gonul Bilgin, N. , & Coşkun, N. (2018). Comparison Result of Some Gadjiev Ibragimov Type Operators. Karaelmas Fen ve Mühendislik Dergisi,8(1), 188-196.

Gonul Bilgin, N. ,& Ozgur, N. (2019). Approximation by Two Dimensional Gadjiev-Ibragimov Type Operators. Ikonion Journal of Mathematics, 1(1), 1-10.

Herdem, S., & Büyükyazıcı, I. (2018). Ibragimov–Gadjiev operators based on q-integers. Advances in Difference Equations, 2018(1), 1-10.

Ispir, N., Aral, A., & Doğru, O. (2008). On Kantorovich process of a sequence of the generalized linear positive operators. Numerical Functional Analysis and Optimization, 29(5-6), 574-589.

Özarslan, M. A., & Aktuğlu, H. (2013). Local approximation properties for certain King type operators. *Filomat*,*27*(1), 173-181.

Ulusoy, G., Deniz, E., & Aral, A. (2015). Simultaneous approximation with generalized Durrmeyer operators. Applied Mathematics and Computation, 260, 126-134.